Tutorial 1 (20 Jan)

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Foreword

⓪ Tutorial notes will be uploaded to the course webpage after tutorials.

Also, tutorials will be recorded.

① Personal Information :

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② Tutorial Arrangement (20 Jan - 24 Feb) :

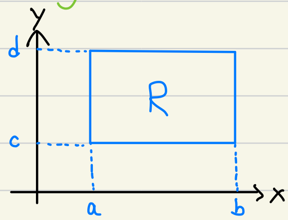
- (1330 - 1355 ($\pm\varepsilon$) / 1530 - 1555 ($\pm\varepsilon$)) Problems and Solutions
- (1355 ($\pm\varepsilon$) - 1415 / 1555 ($\pm\varepsilon$) - 1615) Class Exercises
- (1415 - 1430 / 1615 - 1630) Submission of Class Exercises

③ Reference : [George B. Thomas] Thomas' Calculus

[James Stewart] Calculus - Multivariate Calculus

Fubini 's Theorem

Thm 1 (Fubini's Theorem for continuous functions over rectangles)

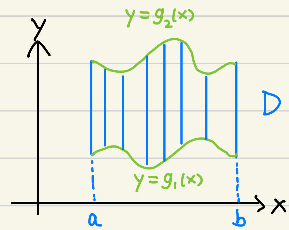


- $R := [a,b] \times [c,d] \subseteq \mathbb{R}^2$: rectangle
- $f : R \to \mathbb{R}$: continuous function

then $\displaystyle\iint_R f\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx = \int_c^d \int_a^b f(x,y)\,dx\,dy.$

Thm 2 (Fubini's Theorem for continuous functions over more general regions)

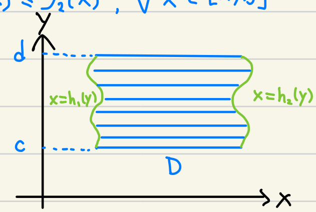
- $D \subseteq \mathbb{R}^2$: bounded region
- $f : D \to \mathbb{R}$: continuous function



(a) If $D = \{(x,y) \in \mathbb{R}^2 \mid a \leq x \leq b\ ;\ g_1(x) \leq y \leq g_2(x)\}$,

where $g_1, g_2 : [a,b] \to \mathbb{R}$ are continuous with $g_1(x) \leq g_2(x)$, $\forall x \in [a,b]$

then $\displaystyle\iint_D f\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx.$



(b) If $D = \{(x,y) \in \mathbb{R}^2 \mid c \leq y \leq d\ ;\ h_1(y) \leq x \leq h_2(y)\}$,

where $h_1, h_2 : [c,d] \to \mathbb{R}$ are continuous with $h_1(y) \leq h_2(y)$, $\forall y \in [c,d]$

then $\displaystyle\iint_D f\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$

(c) If $D = \{a \leq x \leq b\ ;\ g_1(x) \leq y \leq g_2(x)\} = \{c \leq y \leq d\ ;\ h_1(y) \leq x \leq h_2(y)\}$

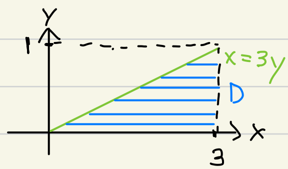
then $\displaystyle\iint_D f\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy.$

Ex) Evaluate $\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy$.

Sol) Idea: Apply Thm 2c to interchange the order of integration.

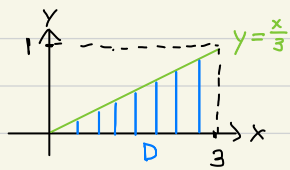
Step 1: Describe the region of integration D using the given order of variables.

$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le y \le 1 \;;\; 3y \le x \le 3\}$



Step 2: Describe D using different order of variables.

$D = \{(x,y) \in \mathbb{R}^2 \mid 0 \le x \le 3 \;;\; 0 \le y \le \frac{x}{3}\}$



Step 3: Compute the integral by interchanging the order of integration using Thm 2c.

$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{\frac{x}{3}} e^{x^2}\,dy\,dx = \int_0^3 \left[y e^{x^2}\right]_0^{\frac{x}{3}} dx$$

$$= \frac{1}{3}\int_0^3 x e^{x^2}\,dx$$

$$= \frac{1}{6}\left[e^{x^2}\right]_0^3$$

$$= \frac{1}{6}\left(e^9 - 1\right)$$